Definition

Ex 4) Prove

DEFINITION --- Dot product

The ***dot product*** or ***inner product*** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is → $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$

Ex 1) Find each dot product:

a) $\langle 3, 4\rangle \cdot \langle 5, 2\rangle$

$3 \cdot 5 + 4 \cdot 2$

23

b) $\langle 1, -2\rangle \cdot \langle -4, 3\rangle$

$1 \cdot -4 + -2 \cdot 3$

-10

c) $(2\mathbf{i} - \mathbf{j}) \cdot (3\mathbf{i} - 5\mathbf{j})$

$\langle 2, -1\rangle \cdot \langle 3, -5\rangle$

$2 \cdot 3 + -1 \cdot -5$

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Properties of the Dot product --------- Let **u**, **v**, and **w** be vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
3. $0 \cdot \mathbf{u} = 0$
4. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
   $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
5. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

Ex 2) Use the dot product to find the length of vector $\mathbf{u} = \langle 4, -3\rangle$

$u \cdot u$

$\langle 4, -3\rangle \cdot \langle 4, -3\rangle$

$16 + 9$

$u \cdot u = |u|^2 = 25$ $\quad |u| = 5$

Angle Between Two Vectors $\quad \cos\theta = \dfrac{u \cdot v}{|u||v|}$ and $\theta = \cos^{-1}\left(\dfrac{u \cdot v}{|u||v|}\right)$

Ex 3) Find the angle between two vectors **u** & **v**.

a) $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle -2, 5\rangle$

$\cos\theta = \dfrac{u \cdot v}{|u| \cdot |v|}$

$\theta = \cos^{-1}\left(\dfrac{2 \cdot -2 + 3 \cdot 5}{\sqrt{13} \cdot \sqrt{29}}\right)$

$\theta = 55.5°$

b) $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle -1, -3\rangle$

$\theta = \cos^{-1}\left(\dfrac{2 \cdot -1 + 1 \cdot -3}{\sqrt{5} \cdot \sqrt{10}}\right)$

$\theta = 135°$

Definition --------- Orthogonal Vectors → The vectors **u** and **v** are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

Ex 4) Prove that the vectors $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle -6, 4\rangle$ are orthogonal.

$$2\cdot -6 + 3\cdot 4$$
$$-12 + 12$$
$$0$$
Orthogonal

Projection of u onto v ------- If **u** and **v** are nonzero vectors, the projection of **u** onto **v** is $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$

Ex 5) Find the vector projection of $\mathbf{u} = \langle 6, 2\rangle$ onto $\mathbf{v} = \langle 5, -5\rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\text{Proj}_v u = \frac{6\cdot 5 + 2\cdot -5}{(\sqrt{50})^2} \cdot \langle 5,-5\rangle$$
$$= \frac{2\cancel{0}}{5\cancel{0}} \langle 5,-5\rangle$$
$$\text{Proj}_v u = \langle 2,-2\rangle$$

① Unknown + $\text{Proj}_v u = u$

Unknown + $\langle 2,-2\rangle = \langle 6,2\rangle$
$-\langle 2,-2\rangle$ $-\langle 2,-2\rangle$

Unknown = $\langle 4,4\rangle$

$\langle 4,4\rangle + \langle 2,-2\rangle = \langle 6,2\rangle$

Ex 6) Find the vector projection of $\mathbf{u} = \langle -3, 4\rangle$ onto $\mathbf{v} = \langle 12, -5\rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\text{Proj}_v u = \frac{-3\cdot 12 + 4\cdot -5}{13^2} \langle 12,-5\rangle$$
$$= \frac{-56}{169} \langle 12,-5\rangle$$
$$= \left\langle -\frac{672}{169}, \frac{280}{169}\right\rangle$$
or
$$\langle 3.98, 1.66\rangle$$

Unknown + $\left\langle -\frac{672}{169}, \frac{280}{169}\right\rangle = \langle -3,4\rangle$

Unknown = $\left\langle \frac{165}{169}, \frac{396}{169}\right\rangle$

$\left\langle \frac{165}{169}, \frac{396}{169}\right\rangle + \left\langle \frac{-672}{169}, \frac{280}{169}\right\rangle = \langle -3,4\rangle$

or

$\langle .98, 2.34\rangle + \langle -3.98, 1.67\rangle = \langle -3,4\rangle$ 8

Unit 7 HW Pg. 9

(1) $u = \langle 3, 9 \rangle$
$v = \langle 6, 5 \rangle$

$18 + 45$

63

(2) $u = -i + 5j$
$v = -6i - 2j$
$\langle -1, 5 \rangle \cdot \langle -6, -2 \rangle$
$6 + -10$
-4

(3) $\langle -7, -9 \rangle \cdot \langle 8 \cdot 4 \rangle$
$-56 + -36$

-92

(4) $\langle 0, 9 \rangle \cdot \langle 5, -8 \rangle$
$0 + -72$
-72

(5) $u = \langle 4, -9 \rangle$
$v = \langle -9, 4 \rangle$

$4 \cdot -9 + -9 \cdot 4$
$\cancel{0}\ -72$
~~Orthogonal~~ Neither

(6) $u = -5i - 2j$
$v = -10i + 25j$
$-5 \cdot -10 + -2 \cdot 25$
0
Orthogonal

(7) $\langle -2, 2 \rangle \langle 8, -1 \rangle$
$\cos\theta = \dfrac{-2 \cdot 8 + 2 \cdot -1}{\sqrt{8} \cdot \sqrt{65}}$

$\theta = 142.1°$

(8) $\langle 3, 6 \rangle \langle 3, -8 \rangle$
$\theta = \cos^{-1}\left(\dfrac{3 \cdot 3 + 6 \cdot -8}{\sqrt{45} \cdot \sqrt{73}}\right)$

$\theta = 132.9°$

(9) $u = \langle -8, -2 \rangle$
$v = \langle -3, 3 \rangle$
$\theta = \cos^{-1}\left(\dfrac{-8 \cdot -3 + -2 \cdot 3}{\sqrt{68} \cdot \sqrt{18}}\right)$
$\theta = 59.04°$

(10) $u = -8j$
$v = -9i - 2j$
$\theta = \cos^{-1}\left(\dfrac{0 \cdot -9 + -8 \cdot -2}{8 \cdot \sqrt{85}}\right)$

$\theta = 77.5°$

(11) $u = \langle 8, 2 \rangle$
$v = \langle -7, -3 \rangle$
$\text{Proj}_v u = \dfrac{8 \cdot -7 + 2 \cdot -3}{58} \langle -7, -3 \rangle$

$= \left\langle \dfrac{217}{29}, \dfrac{93}{29} \right\rangle$

(12) $u = 5i - 5j$
$v = 7i - 5j$
$\text{Proj}_v u = \dfrac{5 \cdot 7 + -5 \cdot -5}{74} \langle 7, -5 \rangle$

$= \left\langle \dfrac{210}{37}, \dfrac{-150}{37} \right\rangle$

(13) $u = \langle -2, -3 \rangle$
$v = \langle -7, 9 \rangle$
$\text{Proj}_v u = \dfrac{-2 \cdot -7 + -3 \cdot 9}{130} \langle -7, 9 \rangle$

$\text{Proj}_v u = \left\langle \dfrac{7}{10}, \dfrac{-9}{10} \right\rangle$

Unknown $+ \left\langle \dfrac{7}{10}, \dfrac{-9}{10} \right\rangle = \langle -2, -3 \rangle$

$\left\langle \dfrac{-27}{10}, \dfrac{-21}{10} \right\rangle + \left\langle \dfrac{7}{10}, \dfrac{-9}{10} \right\rangle = \langle -2, -3 \rangle$

Practice with Dot Products and Angles between Vectors

1) Find the dot product of **u** and **v**. Then determine if it is orthogonal.

a) $u = \langle 3,-2\rangle$ *and* $v = \langle -5,1\rangle$

$-15 + -2$

-17

b) $u = \langle -2,-3\rangle$ *and* $v = \langle 9,-6\rangle$

$-18 + 18$

0

c) $u = \langle -3,4\rangle$ *and* $v = \langle 3,6\rangle$

$-9 + 24$

15

d) $u = \langle 2,7\rangle$ *and* $v = \langle -14,4\rangle$

$-28 + 28$

0

2) Find the magnitude of:

a) $c = \langle -1,-7\rangle$

$\sqrt{50}$

b) $a = \langle -6,5\rangle$

$\sqrt{61}$

c) $m = \langle -3,11\rangle$

$\sqrt{130}$

3) Find the angle θ between vectors **u** and **v** to the nearest tenth of a degree.

a) $u = \langle -5,-2\rangle$ *and* $v = \langle 4,4\rangle$

$\theta = \cos^{-1}\left(\frac{-5\cdot 4 + -2\cdot 4}{\sqrt{29}\cdot\sqrt{32}}\right)$

$\theta = 156.8^\circ$

b) $u = \langle 9,5\rangle$ *and* $v = \langle -6,7\rangle$

$\theta = \cos^{-1}\left(\frac{9\cdot -6 + 5\cdot 7}{\sqrt{106}\cdot\sqrt{85}}\right)$

$\theta = 101.5^\circ$

c) $u = \langle -3,-5\rangle$ *and* $v = \langle 2,-3\rangle$

$\theta = \cos^{-1}\left(\frac{-3\cdot 2 + -5\cdot -3}{\sqrt{34}\cdot\sqrt{13}}\right)$

$\theta = 64.7^\circ$

d) $u = \langle 1,-4\rangle$ *and* $v = \langle 2,6\rangle$

$\theta = \cos^{-1}\left(\frac{1\cdot 2 + -4\cdot 6}{\sqrt{17}\cdot\sqrt{40}}\right)$

$\theta = 147.5^\circ$

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