Precalculus Unit 7 — Notes—Parametric Equations and Motion

DEFINITION---The graph of the ordered pairs (x, y) where: $x = f(t)$, $y = g(t)$ are functions defined on an interval I of t-values is a PARAMETRIC CURVE. The equations are PARAMETRIC EQUATIONS for the curve, the variable t is a PARAMETER, and I is the PARAMETER INTERVAL.

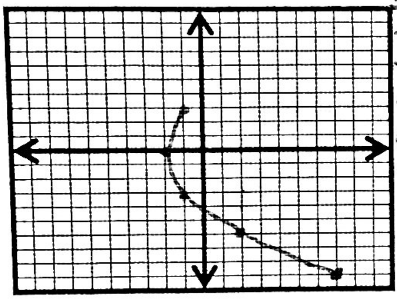
*When we give parametric equations and a parameter interval for a curve, we have "**parametrized**" the curve.

A "parametrization**" of a curve consists of the parametric equations AND the interval of t-values.

EXAMPLE 1 Graphing Parametric Equations

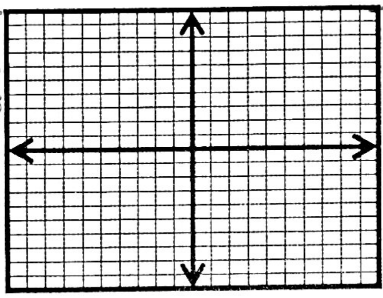
For the given parameter interval, graph the parametric equations using your graphing calculator & note the difference you see: $x = t^2 - 2$, $y = 3t$

(a) $-3 \le t \le 1$



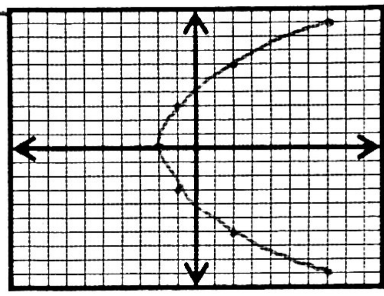
t	x	y
-3	7	-9
-2	2	-6
-1	-1	-3
0	-2	0
1	-1	3

(b) $-2 \le t \le 3$



t	x	y
-2		
-1		
0		
1		
2		
3		

(c) $-3 \le t \le 3$



t	x	y
-3	7	[illegible]
-2	2	[illegible]
-1	-1	[illegible]
0	-2	[illegible]
1	-1	3
2	2	6
3	7	9

*What do you think is the point of doing this example?

If we do not specify a parameter interval for the parametric equations $x = f(t)$, $y = g(t)$, it is understood that the parameter t can take on all values which produce real numbers for x and y.

EXAMPLE 2 Eliminating the Parameter

Eliminate the parameter and identify the graph of the parametric curve $x = 1 - 2t$, $y = 2 - t$, $-\infty \le t \le \infty$.
(without calculator)

$t = 2 - y$

$x = 1 - 2(2 - y)$

$x = 1 - 4 + 2y$

$2y = x + 3$

$y = \frac{1}{2}x + \frac{3}{2}$ linear

EXAMPLE 3 Eliminating the Parameter

Eliminate the parameter and identify the graph of the parametric curve: $x = t^2 - 2$, $y = 3t$.
(without a calculator)

$t = \frac{y}{3}$

$x = \left(\frac{y}{3}\right)^2 - 2$

$x = \frac{y^2}{9} - 2$

$x + 2 = \frac{y^2}{9}$

$y^2 = 9x + 18$

$y = \pm\sqrt{9x + 18}$ sideways parabola

EXAMPLE 4 Eliminating the Parameter

Eliminate the parameter and identify the graph of the parametric curve: $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

$\frac{x}{2} = \cos t$ $\quad \frac{y}{2} = \sin t$

$\left(\frac{x}{2}\right)^2 = \cos^2 t$ $\quad \left(\frac{y}{2}\right)^2 = \sin^2 t$

$\cos^2 t + \sin^2 t = 1$

$\frac{x^2}{4} + \frac{y^2}{4} = 1$

$x^2 + y^2 = 4$ circle

$(x-h)^2 + (y-k)^2 = r^2$ $\quad r = 2$

center $(0, 0)$

EXAMPLE 5 Finding Parametric Equations for a Line

Find a parametrization of the line through the points $(-2, 3)$ & $(3, 6)$.

slope: $\frac{6-3}{3-(-2)} = \frac{3}{5}$

$\langle -2, 3\rangle + T\langle 5, 3\rangle$

$\langle -2, 3\rangle + \langle 5T, 3T\rangle$

$\langle -2 + 5T, 3 + 3T\rangle$

$-2 + 5T = -2$ $\quad T = 0$

$-2 + 5T = 3$ $\quad T = 1$

$0 \le T \le 1$

$x = -2 + 5T$

$y = 3 + 3T$

EXAMPLE 6 Simulating Horizontal Motion
Gary walks along a horizontal line (think of it as a number line) with the coordinate of his position (in meters) given by $s = -0.1(t^3 - 20t^2 + 110t - 85)$ where $0 \le t \le 12$. Use parametric equations and a graphing calculator to simulate his motion. Estimate the times when Gary changes direction.

$y = -.1(t^3 - 20t^2 + 110t - 85)$
$x = t$

$T = 3.9$
$T = 9.4$

EXAMPLE 7 Finding Height of a Projectile
A projectile is launched straight up from ground level with an initial velocity of 288 ft/sec.

(a) When is the projectile's height above ground 1152 ft?

6 sec 12 sec

(b) When is the projectile's height above ground at least 1152 ft?

$6 \le T \le 12$

Projectile Motion

Suppose an object is launched vertically from a point s_0 feet above the ground with an initial velocity of v_0 feet per second. The vertical position s (in feet) of the object t seconds after it is launched is

$$s = -16t^2 + v_0t + s_0.$$

$y = -16T^2 + 288T + 0$
$x = T$

EXAMPLE 8 Simulating Projectile Motion
A distress flare is shot straight up from a ship's bridge 75 ft above the water with an initial velocity of 76 ft/sec. Graph the flare's height against time, give the height of the flare above water at each time, and simulate the flare's motion for each length of time.

(a) 1 sec (b) 2 sec (c) 4 sec (d) 5 sec

135 ft 163 ft 123 ft 55 ft

$y = -16T^2 + 76T + 75$
$x = T$

Suppose that a baseball is thrown from a point y_0 feet above ground level with an initial speed of v_0 ft/sec at an angle θ with the horizontal. The initial velocity can be represented by the vector **v** = $\langle v_0 \cos\theta, v_0 \sin_0\theta \rangle$.
The path of the object is modeled by the parametric equations:

x = $V_0(\cos\theta)T$
[The x-component is simply distance = (x-component of initial velocity) × time]

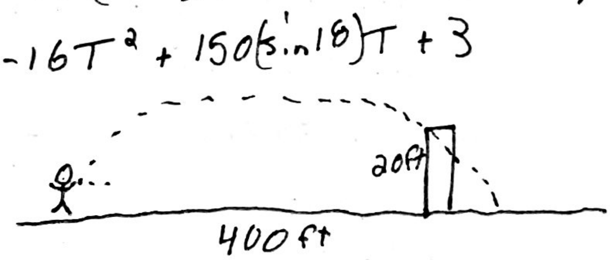
and

y = $-16T^2 + V_0(\sin\theta)T + y_0$
[The y-component is the familiar vertical projectile-motion equation using the y-component of initial velocity vector]

EXAMPLE 9 Hitting a Baseball
Kevin hits a baseball at 3 ft above the ground with an initial speed of 150 ft/sec at an angle of 18° with the horizontal. Will the ball clear a 20-ft wall that is 400 ft away?

$x = 150(\cos 18)T$
$y = -16T^2 + 150(\sin 18)T + 3$

NO



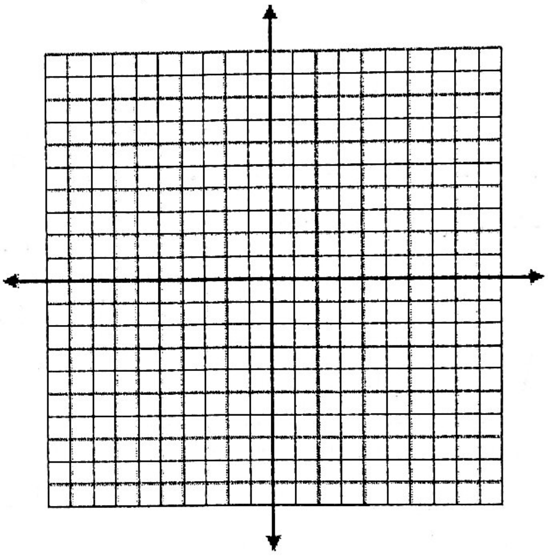
Precalculus Unit 7 *Homework—Parametrics*

1. Fill in the table, plot the points, and sketch the parametric equation for t [-2,6]

$x = \sqrt{t^2 + 1}$

$y = 2 - t$

t	x	y
-2	2.2	4
-1	1.4	3
0	1	2
1	1.4	1
2	2.2	0
3	3.2	-1
4	4.1	-2
5	5.1	-3
6	6.1	-4



In Exercises 2-7, eliminate the parameter t and identify the graph.

2. $x = 3 - 5t, y = 4 + 3t$

$\frac{x-3}{-5} = t$

$y = 4 + 3\left(\frac{x-3}{-5}\right)$

$y = \frac{-3x}{-5} + \frac{29}{5}$

3. $x = 4 + t, y = -8 - 5t, \ -3 \le t \le 5$

$t = x - 4$

$y = -8 - 5(x+4)$

$y = -5x - 12$

4. $x = 2t^2 - 3, y = t - 1$

$t = y + 1$

$x = 2(y+1)^2 - 3$

$\pm\sqrt{\frac{x+3}{2}} = y + 1$

$y = -1 \pm \sqrt{\frac{x+3}{2}}$

5. $x = 3\cos t, y = 3\sin t$

$\frac{x}{3} = \cos t \quad \frac{y}{3} = \sin t$

$\frac{x^2}{9} = \cos^2 t \quad \frac{y^2}{9} = \sin^2 t$

$\cos^2 t + \sin^2 t = 1$

$\frac{x^2}{9} + \frac{y^2}{9} = 1$

$x^2 + y^2 = 9$

6. $x = e^{2t} - 1, y = e^t$

$e^{2t} = x + 1$

$e = \frac{\ln(x+1)}{2}$

$y = e^{\frac{1}{2}\ln(x+1)}$

$y = (x+1)^{\frac{1}{2}} \quad y = \sqrt{x+1}$

7. $x = t^3, y = \ln t, t > 0$

$t = \sqrt[3]{x} \quad y = \ln\sqrt[3]{x}$

In Exercises 8-9, find a parametrization for the curve.

8. The line through the points (–1, –2) & (3, 4)

$m = \frac{4--2}{3--1} = \frac{6}{4} = \frac{3}{2}$

$\langle -1, -2 \rangle + T\langle 2, 3 \rangle$

$\langle -1 + 2T, -2 + 3T \rangle$

$x = 2T - 1$

$y = 3T - 2$

9. The line segment with endpoints (–2, 3) and (5, 1)

$m = \frac{1-3}{5--2} = \frac{-2}{7}$

$\langle -2, 3 \rangle + T\langle 7, -2 \rangle$

$\langle -2 + 7T, 3 - 2T \rangle$

$\boxed{\begin{array}{l} x = 7T - 2 \\ y = -2T + 3 \end{array}}$

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10. Stewart shoots an arrow straight up from the top of a building with initial velocity of 245 ft /sec. The arro leaves from a point 200 ft above level ground.
(a) Write an equation that models the height of the arrow as a function of time t.
(b) Use parametric equations to simulate the height of the arrow.
(c) Use parametric equations to graph height against time.
(d) How high is the arrow after 4 sec?
(e) What is the maximum height of the arrow? When does it reach its maximum height?
(f) How long will it be before the arrow hits the ground?

(a) $h = -16T^2 + 245T + 200$

(b) $y = -16T^2 + 245T + 200$
$x = T$

(c) 

(d) 924 ft.

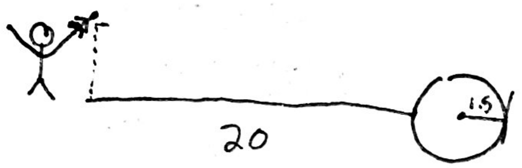
(e) 1137.9ft @ 7.7sec

(f) ≈ 16.1sec

11. Gretta and Lois are launching yard darts 20 ft from the front edge of a circular target of radius 18 in. If Gretta releases the dart 5 ft above the ground with an initial velocity of 20 ft/sec and at a 50° angle with the horizontal, will the dart hit the target?

$x = 20(\cos 50)T$

$y = -16T^2 + 20(\sin 50)T + 5$



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