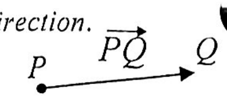
Precalculus Unit 7 Notes—Vectors in the Plane

Quantities that have both direction & magnitude are represented by vectors.

Two vectors are equal if their corresponding directed line segments have the same *length & direction.*
Two vectors are equal if and only if they have the same ***component form.***



DEFINITION --- Component form and Magnitude of a Vector

If v is a vector in the plane equal to the vector with initial point (0, 0) and terminal point (v_1, v_2), then the ***component form*** of v is $$v = \langle v_1, v_2 \rangle$$

Note: *If a vector has initial point* (x_1, y_1) *and terminal point* (x_2, y_2)*, it component form is* $\boxed{\langle x_2 - x_1, y_2 - y_1 \rangle}$.

The ***magnitude*** (or length) of vector $v = \overrightarrow{PQ}$ determined by $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$|v| = \sqrt{v_1^2 + v_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: *If a vector has a length of 0 and no direction, it is the zero vector and is denoted* < 0, 0 >

Ex 1) Find each vector represented by the directed line segment and calculate their magnitudes.

a) Find u, the vector from R = (-4, 2) to S = (-1, 6)

$\langle -1 - -4, 6 - 2 \rangle$

$\langle 3, 4 \rangle$

$|u| = \sqrt{3^2 + 4^2}$

$|u| = 5$

b) Find v, the vector from 0 = (0, 0) to P = (3, 4)

$\langle 3, 4 \rangle$ $|v| = 5$

c) Show that $u = v$.

$\langle 3,4 \rangle = \langle 3,4 \rangle$

Ex 2) Let u be the vector represented by the directed line segment from $R = (7, -3)$ to $S = (4, -5)$, & v the vector from O (0, 0) to $P = (3, 4)$. Prove that $u = v$.

$u: \langle 4-7, -5 - -3 \rangle$ $v: \langle 3, 4 \rangle$

$\langle -3, -2 \rangle$ $u \neq v$

Ex 3) Find the magnitude of vector R = < 4, -2 >.

$|R| = 2\sqrt{5}$

Ex 4) Find the magnitude of vector P = < -3, -5 >.

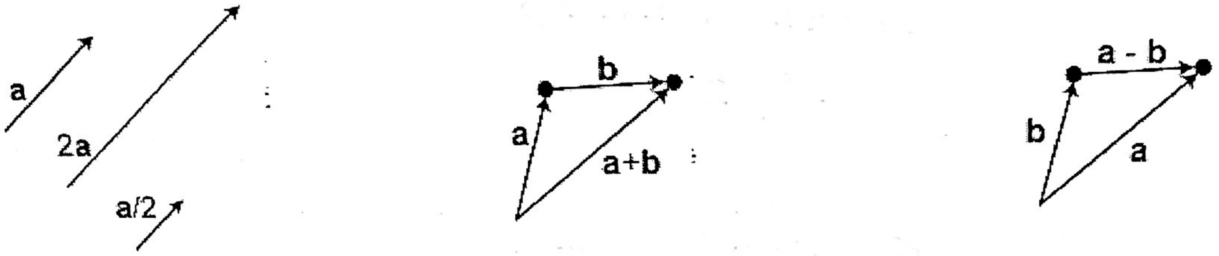
$|P| = \sqrt{34}$

DEFINITION --- Vector Addition and Scalar Multiplication

et $\boldsymbol{u} = \langle u_1, u_2\rangle$ and $\boldsymbol{v} = \langle v_1, v_2\rangle$ be vectors and let k be a real number (scalar).

The **sum** (or ***resultant vector***) of $\boldsymbol{u} + \boldsymbol{v}$ is the vector: $\boldsymbol{u} + \boldsymbol{v} = \langle u_1 + v_1, u_2 + v_2\rangle$

The **scalar product** of vector $\boldsymbol{u}$ and scalar k is the vector: $k\mathbf{u} = k\langle u_1, u_2\rangle = \langle ku_1, ku_2\rangle$



scalar product **vector addition** **vector subtraction**

Ex 5) Let $\boldsymbol{u} = \langle 5, -2\rangle$ and $\boldsymbol{v} = \langle 6, 4\rangle$ find the component form each of the following vectors:

a) $u - v$

$\langle 5-6, -2-4\rangle$

$\langle -1, -6\rangle$

b) $5u$

$5\langle 5, -2\rangle$

$\langle 25, -10\rangle$

c) $3u + (-2)v$

$3\langle 5, -2\rangle + -2\langle 6, 4\rangle$

$\langle 15, -6\rangle + \langle -12, -8\rangle$

$\langle 3, -14\rangle$

DEFINITION --- Unit Vector and the standard Unit Vector

A vector $\boldsymbol{u}$ with length 1 is called a unit vector. To create a unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ simply divide vector $\boldsymbol{v}$ by its magnitude: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{|\mathbf{v}|}\mathbf{v}$

The two unit vectors $\boldsymbol{i} = \langle 1, 0\rangle$ and $\boldsymbol{j} = \langle 0, 1\rangle$ are the **standard unit vectors** and can be used to write a vector as a linear combination of $\boldsymbol{i}$ & $\boldsymbol{j}$.

$|v| = \sqrt{(-3)^2 + (2)^2}$

Ex 6) Find a unit vector in the direction of $\boldsymbol{v} = \langle -3, 2\rangle$, and verify that it has a length equal to 1. Then write the answer in both component form and as a linear combination of the standard unit vectors.

$|v| =$ $\sqrt{13}$ The component form of the vector $\boldsymbol{u}$ a unit vector in the direction of $\boldsymbol{v}$ is $\left\langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$

$\boldsymbol{u}$ written as a linear combination of the standard unit vectors $\boldsymbol{i}$ & $\boldsymbol{j}$ is $\frac{-3}{\sqrt{13}} i + \frac{2}{\sqrt{13}} j$

Ex 7) Find a unit vector in the direction of $\boldsymbol{v} = \langle 5, -3\rangle$, and verify that it has a length equal to 1.

$|v| = \sqrt{5^2 + (-3)^2}$

$|v| = \sqrt{34}$

$\left\langle \frac{5}{\sqrt{34}}, \frac{-3}{\sqrt{34}} \right\rangle$

Unit Vector

$1 = \sqrt{\left(\frac{5}{\sqrt{34}}\right)^2 + \left(\frac{-3}{\sqrt{34}}\right)^2}$

$1 = \sqrt{\frac{25}{34} + \frac{9}{34}}$

$1 = 1$

When 2
both forc

DEFINITION --- Direction Angle

To precisely specify the direction of a vector, state its **direction angle** θ (made by the vector and the positive x-axis)

Using trigonometry, we can see the horizontal component of a vector v is $(|v|\cos\theta)$ and the vertical component is $(|v|\sin\theta)$, thus: $v = (|v|\cos\theta)i + (|v|\sin\theta)j = \langle |v|\cos\theta, |v|\sin\theta\rangle$

Ex 8) Find the components of vector v with direction angle $\theta = 115°$ and magnitude of 6.

$\langle 6\cos 115, 6\sin 115\rangle$

$\langle -2.5, 5.4\rangle$

Ex 9) Find the components of vector v with direction angle $\theta = 230°$ and magnitude of 12.

$\langle 12\cos 230, 12\sin 230\rangle$

$\langle -7.7, -9.2\rangle$

Ex 10) Find the magnitude & direction angle of each vector:

a) $u = \langle 3, 2\rangle$ $|u| = \sqrt{13}$

$3 = \sqrt{13}\cos\theta$, $\frac{3}{\sqrt{13}} = \frac{\sqrt{13}\cos\theta}{\sqrt{13}}$

$\cos\theta = \frac{3}{\sqrt{13}}$

$\theta = 33.7°$

$2 = \sqrt{13}\sin\theta$, $\frac{2}{\sqrt{13}} = \frac{\sqrt{13}\sin\theta}{\sqrt{13}}$

$\theta = \sin^{-1}\left(\frac{2}{\sqrt{13}}\right)$

$\theta = 33.7°$

b) $v = \langle -2, -5\rangle$ $|v| = \sqrt{29}$

$-2 = \sqrt{29}\cos\theta$

$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{29}}\right)$

$\theta = 111.8°$

$-5 = \sqrt{29}\sin\theta$

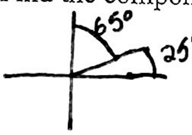
$\theta = \sin^{-1}\left(\frac{-5}{\sqrt{29}}\right)$

$\theta = -68.2°$

$\boxed{\theta = 248.2°}$

- **The *velocity* of a moving object is a vector because it has both magnitude and direction.**
- **The magnitude of velocity is *speed*.**
- **A *bearing* measures clockwise from the north.**

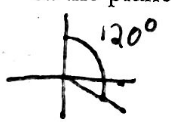
Ex 11) A jet is flying on a bearing of 65° at 500 mph Find the component form of the velocity of the jet.



$\langle 500\cos 25, 500\sin 25\rangle$

$\langle 453.2, 211.3\rangle$

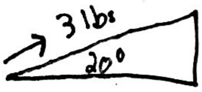
Ex 12) A plane is flying on a bearing of 120° at 425 mph. Find the component form of the velocity of the plane.



$\langle 425\cos(-30), 425\sin(-30)\rangle$

$\langle 368.1, -212.5\rangle$

Ex 13) Suppose we are pushing a crate up a 20° inclined plane with a force of 3.0 lb. Find the component form of the force.



$\langle 3\cos 20, 3\sin 20\rangle$

$\langle 2.8, 1.0\rangle$

When 2 forces act upon an object, their **sum** is the ***resultant*** vector showing the velocity of the object itself with both forces acting upon it.

Ex 14) a) An airplane is flying on a bearing of 80° at 540 mph. Find the component form of the velocity of the airplane.

$\langle 540\cos 10, 540\sin 10 \rangle$

$\langle 531.8, 93.8 \rangle$ Plane

b) Suppose the plane is flying with a 55 mph wind that has a bearing of 100°. What is the actual ground speed and direction of the plane?

Wind

$\langle 55\cos -10, 55\sin -10 \rangle$

$\langle 54.2, -9.6 \rangle$

Plane $\langle 531.8, 93.8 \rangle$

+ Wind $\langle 54.2, -9.6 \rangle$

Result $\langle 586, 84.2 \rangle$

$|R| = 592.0$ mph

$\tan\theta = \frac{84.2}{586}$ $\quad\theta = 8.2°$

$586 = 592\cos\theta$ $\quad\theta = 8.2°$

$84.2 = 592\sin\theta$ $\quad\theta = 8.2°$

Ex 15) A plane is flying on a bearing of 94.14° at 450mph. A wind is blowing on a bearing of 60° at 65 mph. Find the component form of the resultant velocity of the airplane. Then state the actual speed and direction of the plane.

Plane $\langle 450\cos -4.14, 450\sin -4.14 \rangle$

+ wind $\langle 65\cos 30, 65\sin 30 \rangle$

Resultant: $\langle 505.1, .0128 \rangle$

Actual speed: $\sqrt{(505.1)^2 + (.0128)^2} \approx 505.1$ mph

Direction: $\tan\theta = \frac{.0128}{505.1} \approx .0012$

$\theta = 0°$ Bearing 90°

Bearing?

81.8°

Ex 16) An airplane's velocity with no wind is 580 km/h with a bearing of N60°E. The wind at the altitude of the plane has a velocity of 60 km/h and is blowing SE (which means S45°E). What is the true speed and bearing of the plane?

Plane: $\langle 580\cos 30, 580\sin 30 \rangle$

+ Wind: $\langle 60\cos 45, 60\sin 45 \rangle$

Resultant: $\langle 544.7, 332.4 \rangle$

True Speed: $\sqrt{544.7^2 + 332.4^2} \approx 638.1$ mph

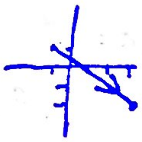
$\tan\theta = \frac{332.4}{544.7}$

$\theta \approx 31.4°$ Bearing: 58.6°

Hw Unit 7 Pg 5

(1)

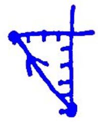
(a)



$v = \langle 4, -3 \rangle$

$|v| = 5$

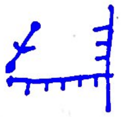
(b)



$v = \langle -4, 5 \rangle$

$|v| = \sqrt{41}$

(c)



$v = \langle -1, -2 \rangle$

$|v| = 5$

(2) (a) v $(5,3)\ (-2,2)$

w $(7,-1)\ (0,-2)$

$v = \langle -7, -1 \rangle$

$w = \langle -7, -1 \rangle$

Equivalent

(b) v $(-10,-3)\ (-1,-12)$

w $(7,-1)\ (-2,8)$

$v = \langle 9, -9 \rangle$

$w = \langle -9, 9 \rangle$

Not

(3) $u = \langle -1, 7 \rangle$ $v = \langle 3, -1 \rangle$

(a) $u + v : \langle 2, 6 \rangle$

(b) $u - v : \langle -4, 8 \rangle$

(c) $4u - 3v : \langle -4, 28 \rangle - \langle 9, -3 \rangle$

$: \langle -13, 31 \rangle$

(d) $u \cdot v$ $-1 \cdot 3 + 7 \cdot -1$

-10

(4) (a) $v = \langle 8, -15 \rangle$

$|v| = 17$

$\langle \frac{8}{17}, \frac{-15}{17} \rangle$

(b) $v = \langle 3, 0 \rangle$

$|v| = 3$

$\langle 1, 0 \rangle$

(c) $v = \langle -4\sqrt{2}, -2 \rangle$

$|v| = 6$

$\langle -\frac{2\sqrt{2}}{3}, -\frac{1}{3} \rangle$

(5) $u : (1,-8)\ (-1,-5) : \langle -2, 3 \rangle$

$v : 3i - 4j \quad : \langle 3, -4 \rangle$

(a) $-2u$

$\langle 4, -6 \rangle$

$4i - 6j$

(b) $u - 2v$

$\langle -2, 3 \rangle - \langle 6, -8 \rangle$

$\cancel{\langle -4, -8 \rangle}$ $\langle -8, 11 \rangle$

$\cancel{\langle -4i - 5j \rangle}$

$-8i + 11j$

(c) $\frac{u}{|v|}$

$\langle -\frac{2}{5}, \frac{3}{5} \rangle$

$-\frac{2}{5}i + \frac{3}{5}j$

(6) (a) $|v| = 6 \quad \theta = 45^\circ$

$\langle 6\cos 45, 6\sin 45 \rangle$

$\langle 3\sqrt{2}, 3\sqrt{2} \rangle$

(b) $|v| = 12 \quad \theta = 240^\circ$

$\langle 12\cos 240, 12\sin 240 \rangle$

$\langle -6, -6\sqrt{3} \rangle$

(c) $|v| = 10 \quad \theta = 6i - 2j$

$\tan\theta = \frac{-2}{6}$

$\theta = -18.4$

$\langle 10\cos -18.4, 10\sin -18.4 \rangle$

(7)

(9)

(a) $\langle 360\cos 185, 360\sin 185 \rangle$

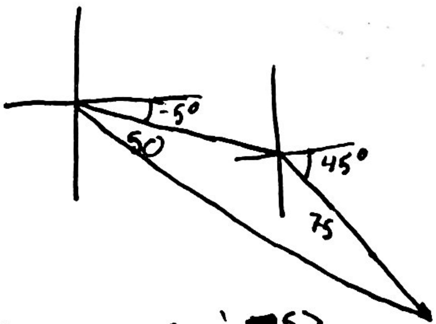
(b) $\langle 38\cos 130, 38\sin 130 \rangle$

(c) $\langle -350.7, 181.3 \rangle$

(d) speed: 394.8 mph

$\theta = 152.7^\circ$

(8)



$\langle 50\cos -5, 50\sin -5 \rangle$

$+ \langle 75\cos -45, 75\sin -45 \rangle$

$\langle 102.8, -57.4 \rangle$

117.7 miles

$\theta = -29.2^\circ$

Bearing 119.2°