Precalculus Unit 1

Notes—Implicit Functions and Inverses

When dealing with a **RELATION** that is not a function, it is often possible to solve for y. Then you can identify the functions which are **IMPLICITLY** defined by the original relation.

***HINT: SOLVE FOR Y. If the equation starts with y^2 when solving you will always end with ± some expression. The positive (+) is one equation, and the negative (-) is the other. These two different equations are the two implicitly defined equations for the given relation.

Example 1 Find two functions defined implicitly by each given relation.

a) $\sqrt{x}=\sqrt{y^2}$

$y=\pm\sqrt{x}$

b) $x^2+2xy+y^2=1$

$(x+y)(x+y)=1$

$\sqrt{(x+y)^2}=\sqrt{1}$

$x+y=\pm 1$

$-x \qquad -x$

$\rightarrow y=-x\pm 1$

(1) $y=-x+1$

(2) $y=-x-1$

c) $9x^2-12xy+4y^2=16$

$(3x-2y)(3x-2y)=16$

$\sqrt{(3x-2y)^2}=\sqrt{16}$

$3x-2y=\pm 4$

$-3x \qquad -3x$

$\frac{-2y}{-2}=\frac{-3x}{-2}\pm\frac{4}{-2}$

$y=\frac{3}{2}x\pm 2$

(1) $y=\frac{3}{2}x+2$

(2) $y=\frac{3}{2}x-2$

d) $\frac{7x^2 - 14xy - 63}{7} = \frac{-7y^2}{7}$

$x^2-2xy+y^2=9$

$(x-y)(x-y)=9$

$\sqrt{(x-y)^2}=\sqrt{9}$

$x-y=\pm 3$

$-y=-x\pm 3$

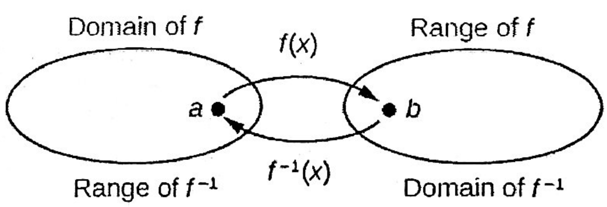
$y=x\pm 3$

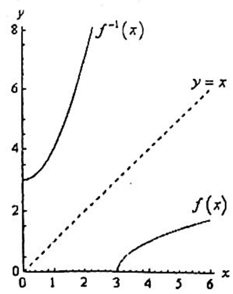
(1) $y=x+3$

(2) $y=x-3$

Inverse Functions & Relations

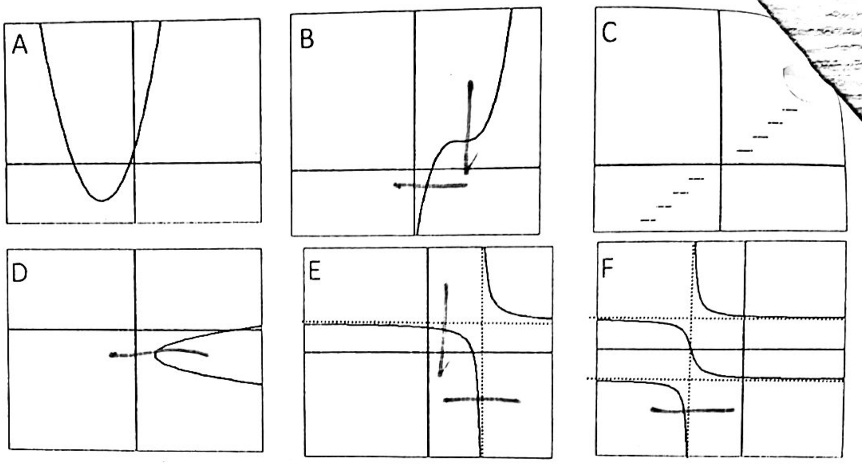
- The most important thing to remember about INVERSES is that x & y switch.
- The inverse of $f(x)$ is denoted $f^{-1}(x)$
- If $f(x)$ & $g(x)$ are inverses of one another then the domain of one is the range of the other & vice-versa.
- A relation is a function if it passes the Vertical Line Test (VLT)
- ★ A relation has an inverse that is a function if it passes the Horizontal Line Test (HLT)
- ★ A function has an inverse function if it is a one-to-one function (meaning it passes both the HLT & VLT)
- The graphical relationship between inverses is that they are reflections of one another over the line $y=x$





Using the Vertical & Horizontal Line Tests

Example 2



(a) Which of the relations to the right are functions?

A, B, C, E

(b) Which of the relations to the right have an inverse function?

B, D, E, F

(c) Which of the relations are one-to-one functions?

B + E

Calculating Inverses Algebraically

Example 3 Given the function $f(x)$ calculate $f^{-1}(x)$ and identify the domain and range of each:

(a) $f(x) = 3\sqrt{x} - 5$ $\qquad f^{-1}(x) = \dfrac{(x+5)^2}{9}$

Domain of $f(x)$: $[0, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $[-5, \infty)$

Range of $f(x)$: $[-5, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $[0, \infty)$

(1) $x = 3\sqrt{y} - 5$
$+5 \qquad +5$

(2) $\dfrac{x+5}{3} = \dfrac{3\sqrt{y}}{3}$

(3) $(\sqrt{y})^2 = \left(\dfrac{x+5}{3}\right)^2$

$y = \dfrac{(x+5)^2}{9}$

(b) $f(x) = \dfrac{x-4}{x+3}$ $\qquad f^{-1}(x) = \dfrac{3x+4}{1-x}$

Domain of $f(x)$: $(-\infty, -3) \cup (-3, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $(-\infty, 1) \cup (1, \infty)$

Range of $f(x)$: $(-\infty, 1) \cup (1, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $(-\infty, -3) \cup (-3, \infty)$

(1) $(y+3)x = \dfrac{y-4}{y+3}(y+3)$

(2) $xy + 3x = y - 4$
$-xy \quad +4 \qquad -xy + 4$

(3) $3x + 4 = y - xy$

(4) $\dfrac{3x+4}{1-x} = \dfrac{y(1-x)}{1-x}$

(5) $y = \dfrac{3x+4}{1-x}$

(c) $f(x) = -(x+1)^3 - 5$ $f^{-1}(x) =$ $\sqrt[3]{-x-5} - 1$

① $x = -(y+1)^3 - 5$
② $x + 5 = -(y+1)^3$
③ $\sqrt[3]{-x-5} = \sqrt[3]{(y+1)^3}$
$y + 1 = \sqrt[3]{-x-5}$
$-1 \qquad -1$
$y = \sqrt[3]{-x-5} - 1$

Domain of $f(x)$: $(-\infty, \infty)$ Domain of $f^{-1}(x)$: $(-\infty, \infty)$

Range of $f(x)$: $(-\infty, \infty)$ Range of $f^{-1}(x)$: $(-\infty, \infty)$

(d) $f(x) = \dfrac{3}{x-5}$ $f^{-1}(x) =$ $\dfrac{3}{x} + 5$ or $\dfrac{5x+3}{x}$

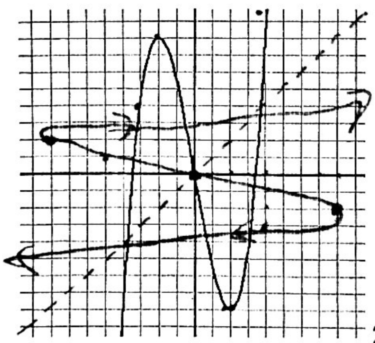
$x = \dfrac{3}{y-5}$

Domain of $f(x)$: $(-\infty, 5) \cup (5, \infty)$ Domain of $f^{-1}(x)$: $(-\infty, 0) \cup (0, \infty)$

Range of $f(x)$: $(-\infty, 0) \cup (0, \infty)$ Range of $f^{-1}(x)$: $(-\infty, 5) \cup (5, \infty)$

Sketching an Inverse Relation From a Graph

Example 4 Given the function $f(x)$ below, sketch $f^{-1}(x)$ and identify the domain and range of both.



D of $f(x)$: $(-\infty, \infty)$ R of $f(x)$: $(-\infty, \infty)$

D of $f^{-1}(x)$: $(-\infty, \infty)$ R of $f^{-1}(x)$: $(-\infty, \infty)$

$f(x)$

0	0
-2	8
2	-8
$\approx$ 1	-5
-3	4

$\rightarrow$ $f^{-1}(x)$

0	0
8	-2
-8	2
-5	1
4	-3

The Inverse Composition Rule

A function f is one-to-one with inverse function g if and only if $f(g(x)) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f.

Verifying Inverses

Example 5 Given the two functions below, verify that they are inverses of one another.

(a) $f(x)=-\frac{1}{2}(x+3)^2-4$ & $g(x)=\sqrt{-2x-8}-3$

$f(g(x))=-\frac{1}{2}\left((\sqrt{-2x-8}-3)+3\right)^2-4$

$f(g(x))=-\frac{1}{2}\left(\sqrt{-2x-8}\right)^2-4$

$\quad\quad=-\frac{1}{2}(-2x-8)-4$

$\quad\quad=x+4-4$

$f(g(x))=x$

$g(f(x))=\sqrt{-2\left(-\frac{1}{2}(x+3)^2-4\right)-8}-3$

$=\sqrt{(x+3)^2+8-8}-3$

$=\sqrt{(x+3)^2}-3$

$=x+3-3$

$g(f(x))=x$

(b) $f(x)=\frac{x-11}{3}$ & $g(x)=3x+11$

$f(g(x))=\frac{(3x+11)-11}{3}$

$=\frac{3x}{3}$

$f(g(x))=x$

$g(f(x))=3\left(\frac{x-11}{3}\right)+11$

$=x-11+11$

$g(f(x))=x$

(c) $f(x)=5\sqrt[3]{x}-7$ & $g(x)=\frac{(x+7)^3}{125}$

$f(g(x))=5\sqrt[3]{\frac{(x+7)^3}{125}}-7$

$=5\left(\frac{x+7}{5}\right)-7$

$=x+7-7$

$f(g(x))=x$

$g(f(x))=\frac{[(5\sqrt[3]{x}-7)+7]^3}{125}$

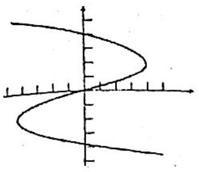
$=\frac{[5\sqrt[3]{x}]^3}{125}$

$=\frac{125x}{125}$

$g(f(x))=x$

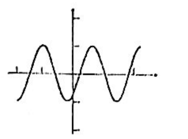
Precalculus Unit 1 *Homework—Implicit Functions and Inverses*

For #1-4, the graph of a relation is shown. (A) Is the relation a function? (B) Does the relation have an inverse that is a function?

1. 

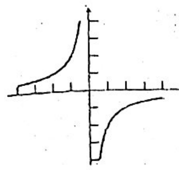
a No

b yes

2. 

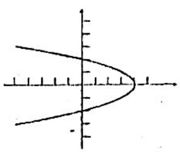
a yes

b No

3. 

a yes

b yes

4. 

a No

b yes

For #5-9, find $f^{-1}(x)$. Give the domain of f^{-1}, including any restrictions "inherited" from f.

5. $f(x) = 3x - 6$

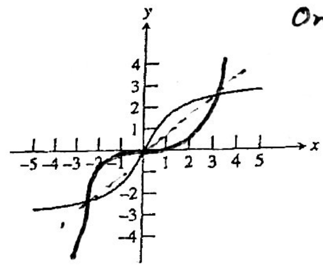
6. $f(x) = \dfrac{2x-3}{x+1}$

7. $f(x) = \sqrt{x-3}$

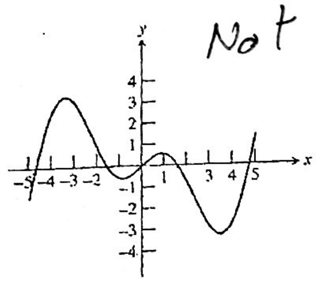
8. $f(x) = x^3$

9. $(x) = \sqrt[3]{x+5}$

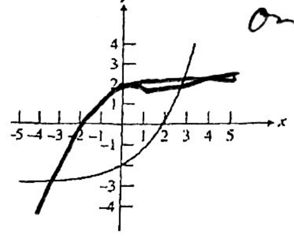
For #10-13, determine whether the function if one-to-one. If it one-to-one, sketch the graph of the inverse.

10. 

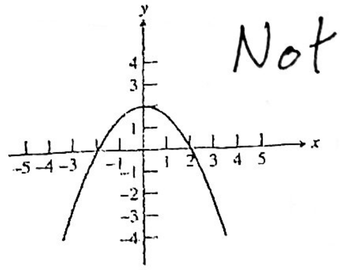
One to One

11. 

Not

12. 

one to one

13. 

Not

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(5) $f(x)=3x-6$

$x=3y-6$

$x+6=3y$

$y=\frac{1}{3}x+2$

$f^{-1}(x)$: D $(-\infty,\infty)$

(6) $f(x)=\frac{2x-3}{x+1}$

$x=\frac{2y-3}{y+1}$

$xy+x=2y-3$

$xy-2y=-x-3$

$y=\frac{-x-3}{x-2}$

D: $(-\infty,2)\cup(2,\infty)$

(7) $f(x)=\sqrt{x-3}$

$x=\sqrt{y-3}$

$x^2+3=y$

D: $[3,\infty)$

(8) $f(x)=x^3$

$x=y^3$

$y=\sqrt[3]{x}$

D: $(-\infty,\infty)$

(9) $f(x)=\sqrt[3]{x+5}$

$x=\sqrt[3]{y+5}$

$y=x^3-5$

D: $(-\infty,\infty)$

(14) $f(x)=3x-2$ $g(x)=\frac{x+2}{3}$

$f(g(x))=3\left(\frac{x+2}{3}\right)-2$

$=x$

$g(f(x))=\frac{3x-2+2}{3}$

$=x$

(15) $f(x)=x^3+1$ $g(x)=\sqrt[3]{x-1}$

$f(g(x))=\left(\sqrt[3]{x-1}\right)^3+1$

$=x$

$g(f(x))=\sqrt[3]{x^3+1-1}$

$=x$

(16) $f(x)=\frac{7}{x}$ $g(x)=\frac{7}{x}$

$f(g(x))=\frac{7}{\frac{7}{x}}$

$=x$

$g(f(x))=\frac{7}{\frac{7}{x}}$

$=x$

(17) $f(x)=3x-7$

1 to 1

(18) $f(x)=\frac{1}{x-2}$

1 to 1

(19) $f(x)=x^2-9$

Not

(20) $f(x)=x^2+4$

Not

(21) $f(x)=\sqrt{x}$

1 to 1

(22) $f(x)=\sqrt[3]{x}$

1 to 1

(23) $f(x)=|x|$

Not

(24) $f(x)=3$

Not

(25) $f(x)=\sqrt{4-x^2}$

Not

(26) $f(x)=2x^3-4$

1 to 1

(27) $f(x)=x^2-9$

Not

(28) $f(x)=\frac{1}{x^2}$

Not

For #14-16, confirm that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

14. $f(x) = 3x - 2$ and $g(x) = \frac{x+2}{3}$

15. $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$

16. $f(x) = \frac{7}{x}$ and $g(x) = \frac{7}{x}$

For #17-28, determine whether the function is one-to-one.

17. $f(x) = 3x - 7$

18. $f(x) = \frac{1}{x-2}$

19. $f(x) = x^2 - 9$

20. $f(x) = x^2 + 4$

21. $f(x) = \sqrt{x}$

22. $f(x) = \sqrt[3]{x}$

23. $f(x) = |x|$

24. $f(x) = 3$

25. $f(x) = \sqrt{4 - x^2}$

26. $f(x) = 2x^3 - 4$

27. $f(x) = x^2 - 9$

28. $f(x) = \frac{1}{x^2}$

For #29-31, find two functions defined implicitly by each given relation.

29. $x^2 + y^2 = 25$ $\longrightarrow$ $y^2 = -x^2 + 25$

$\boxed{y = \pm\sqrt{-x^2 + 25}}$

30. $3x^2 - y^2 = 25$ $\sqrt{y^2} = \sqrt{3x^2 - 25}$

$\boxed{y = \pm\sqrt{3x^2 - 25}}$

31. $3x^2 + 75y^2 = 27 - 30xy$

$\frac{3x^2 + 30xy + 75y^2}{3} = \frac{27}{3}$

$x^2 + 10xy + 25y^2 = 9$

$\sqrt{(x+5y)^2} = \sqrt{9}$

$x + 5y = \pm 3$

$\boxed{y = \frac{-x \pm 3}{5}}$