

Process for Graphing a Rational Function

- 1) Find the x- and y-intercepts, if there are any. The x-intercepts are the zeros of the numerator (after simplifying). The y-intercept is found by substituting zero in for x.
- 2) Find the vertical asymptotes by setting the denominator equal to zero and solving. Be sure to check for removable discontinuities (holes).
- 3) Find the horizontal asymptote, if it exists, by comparing the degrees of the numerator and denominator.

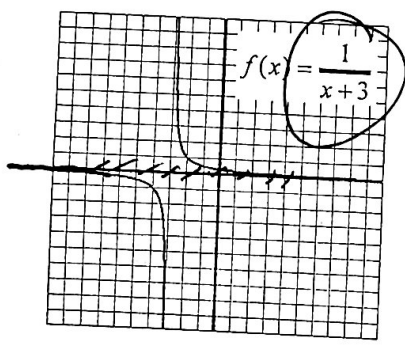
$\frac{\text{little}}{\text{big}} \ni y=0$ When the degree of the numerator is less than the degree of the denominator, the horizontal asymptote will be at $y=0$.
 $\frac{\text{Same}}{\text{Same}} \ni y = \frac{\text{L.C.}}{\text{L.C.}}$ When the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote will be at $y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$

$\frac{\text{Big}}{\text{Little}} \ni \text{No HA, but SA}$ When the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote but there is an oblique asymptote.

- 4) The vertical asymptotes will divide the coordinate grid into regions. In each region graph at least one point. This point will tell us whether the graph will be above or below the horizontal asymptote. Get several points to determine the general shape of the graph.
- 5) Sketch the graph.

Example 1

Find the domain and range of $f(x)$. Identify any vertical or horizontal asymptotes. Use limits to describe its end behavior and its behavior around the vertical asymptotes. Describe the transformations on its parent function.



$f(x) = \frac{1}{x+3}$
 VA: $x = -3$
 HA: $y = 0$
 $\lim_{x \rightarrow -3^+} f(x) = +\infty$
 $\lim_{x \rightarrow -3^-} f(x) = -\infty$
 $\lim_{x \rightarrow -\infty} f(x) = 0$
 $\lim_{x \rightarrow \infty} f(x) = 0$
 $\lim_{x \rightarrow -3} f(x) = \text{D.N.E.}$
 Transf.: $\leftarrow 3$

Example 2

Find the end behavior asymptote for each rational function. HA.

A. $y = \frac{x^2 - 3x + 5}{2x^2 - 1}$ $y = 0$

B. $y = \frac{3x^2 - 2x + 7}{2x^2 - 1}$ $y = \frac{3}{2}$

$$\begin{array}{r} -2 \overline{) 1 \ 3 \ 5} \\ \underline{-2 \ -2} \\ 1 \ 1 \ 13 \end{array}$$

C. $y = \frac{5x^2 - x + 9}{2x^2 - 3x + 4}$ $y = \frac{5}{2}$

D. $y = \frac{x^2 + 3x + 5}{x + 2}$ HA: None
SA: $y = x + 1$

$$y = x + 1 + \frac{3}{x+2}$$

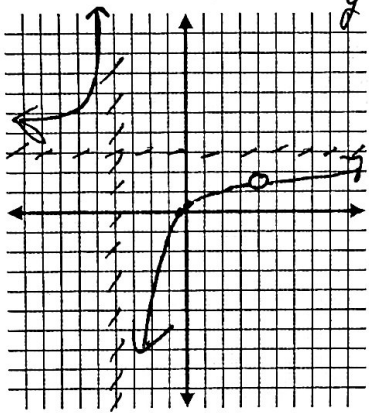
E. $y = \frac{x^0}{2x^0 - 1} \quad y = 0$

F. $y = \frac{x^3 + 2x^2 - x + 5}{x - 1}$ HA: None
SA: $y = x^2 + 3x + 2$

$$\begin{array}{r} 1 \quad 2 \quad -1 \quad 5 \\ 1 \quad 3 \quad 2 \quad 7 \end{array}$$

Example 3 Sketch a graph for each rational function.

A. $f(x) = \frac{3x^2 - 11x - 4}{x^2 - 16} = \frac{(3x+1)(x-4)}{(x+4)(x-4)}$
 $y = \frac{3x+1}{x+4} : \frac{3(4)+1}{4+4}$



Holes: $(4, \frac{13}{8})$

x-ints: $(-\frac{1}{3}, 0)$

y-int: $(0, \frac{1}{4})$

VA: $x = -4$

$(-5, 14)$
 $y = \frac{3(-5)+1}{-5+4} = \frac{-14}{-1} = 14$

EBA: $y = 3$
SA or HA

$\lim_{x \rightarrow -\infty} f(x) = 3$

$\lim_{x \rightarrow \infty} f(x) = 3$

$\lim_{x \rightarrow -4^-} f(x) = \infty$

$\lim_{x \rightarrow -4^+} f(x) = \frac{13}{8}$

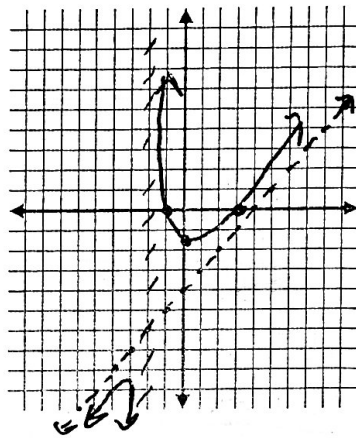
$\lim_{x \rightarrow -4^+} f(x) = -\infty$

$\lim_{x \rightarrow -4^-} f(x) = \frac{13}{8}$

$\lim_{x \rightarrow -4} f(x)$ Does Not Exist

$\lim_{x \rightarrow 4} f(x) = \frac{13}{8}$

B. $f(x) = \frac{x^2 - 2x - 3}{x + 2} = \frac{(x-3)(x+1)}{x+2}$
 $y = \frac{(x-3)(x+1)}{x+2}$



Holes: None

x-ints: $(3, 0)$ $(-1, 0)$

y-int: $(0, -\frac{3}{2})$

VA: $x = -2$

$$\begin{array}{r} -2 \quad 1 \quad -2 \quad -3 \\ 1 \quad -4 \quad 5 \end{array}$$

EBA: HA: None

SA: $y = x - 4$
Doesn't cross SA

$(-3, -12)$
 $y = \frac{(-3-3)(-3+1)}{-3+2} = \frac{12}{-1} = -12$

$\lim_{x \rightarrow -\infty} f(x) = -\infty$

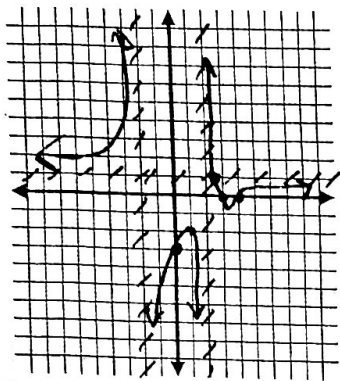
$\lim_{x \rightarrow \infty} f(x) = \infty$

$\lim_{x \rightarrow -2^+} f(x) = \infty$

$\lim_{x \rightarrow -2^-} f(x) = -\infty$

$\lim_{x \rightarrow -2} f(x)$ Does Not Exist

C. $f(x) = \frac{x^2 - 7x + 12}{x^2 - 4} : \frac{(x-4)(x-3)}{(x+2)(x-2)}$



Holes: None

x-ints: (4, 0) (3, 0)

y-int: (0, -3)

VA: x = 2 x = -2

Cross HA? $(\frac{16}{7}, 1)$

$$1 = \frac{x^2 - 7x + 12}{x^2 - 4}$$

$$x^2 - 4 = x^2 - 7x + 12$$

$$-4 = -7x + 12$$

$$-12 \quad -12$$

$$\frac{-16}{-7} = \frac{-7x}{-7}$$

$$x = \frac{16}{7}$$

EBA: y = 1

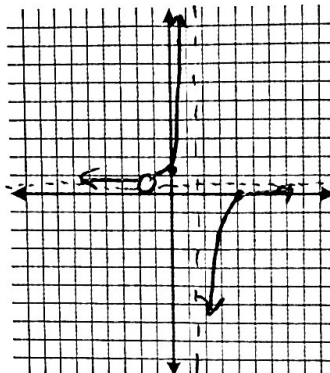
$(5, \frac{2}{21})$

$$\frac{(5-4)(5-3)}{(5+2)(5-2)} = \frac{2}{21}$$

$(1, -2)$

$$\frac{(1-4)(1-3)}{(1+2)(1-2)} = \frac{-3 \cdot -2}{3 \cdot -1} = -2$$

D. $f(x) = \frac{2x^2 - 5x - 12}{4x^2 - 9} : \frac{(2x+3)(x-4)}{(2x+3)(2x-3)}$



Holes: $(-\frac{3}{2}, \frac{11}{12})$

x-ints: 4, 0

y-int: (0, 4/3)

VA: x = 3/2

Hole: $\frac{-\frac{3}{2} - 4}{2(-\frac{3}{2}) - 3} = \frac{-\frac{11}{2}}{-\frac{3}{1} - 3} = \frac{-\frac{11}{2}}{-6} = \frac{11}{12}$

EBA: y = 1/2

Check HA

$$\frac{1}{2} = \frac{x-4}{2x-3}$$

$$\frac{2x-3}{2} = x-4$$

$$x - \frac{3}{2} = x - 4$$

NO

$$\lim_{x \rightarrow -\infty} f(x) = 1$$

$$\lim_{x \rightarrow \infty} f(x) = 1$$

$$\lim_{x \rightarrow -\infty} f(x) = \frac{1}{2}$$

$$\lim_{x \rightarrow \infty} f(x) = \frac{1}{2}$$

$$\lim_{x \rightarrow 2^-} f(x) = \infty$$

$$\lim_{x \rightarrow 2^+} f(x) = -\infty$$

$$\lim_{x \rightarrow -3/2^-} f(x) = \frac{11}{12}$$

$$\lim_{x \rightarrow 3/2^-} f(x) = \infty$$

$$\lim_{x \rightarrow 2^+} f(x) = -\infty$$

$$\lim_{x \rightarrow 2^-} f(x) = \infty$$

$$\lim_{x \rightarrow -3/2^+} f(x) = \frac{11}{12}$$

$$\lim_{x \rightarrow 3/2^+} f(x) = -\infty$$

$$\lim_{x \rightarrow 2} f(x) = \text{Does Not Exist}$$

$$\lim_{x \rightarrow 2} f(x) = \text{Does Not Exist}$$

$$\lim_{x \rightarrow -3/2} f(x) = \frac{11}{12}$$

$$\lim_{x \rightarrow 3/2} f(x) = \text{Does Not Exist}$$

Precalculus Unit 3 Homework—Graphing Rational Equations

For each function, use the points of discontinuity, holes, and asymptotes to sketch a graph. Determine the end behavior and behavior around asymptotes.

1. $f(x) = \frac{5}{x-2}$

Hole(s): None

x-int(s): None

y-int: $(0, -\frac{5}{2})$

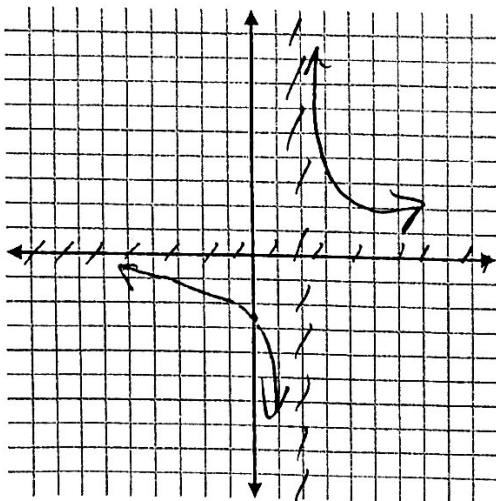
Equations of

ALL asymptotes: $x=2$ $y=0$

$\lim_{x \rightarrow -\infty} f(x) = 0$ $\lim_{x \rightarrow \infty} f(x) = 0$

$\lim_{x \rightarrow 2^-} f(x) = -\infty$ $\lim_{x \rightarrow 2^+} f(x) = \infty$

$\lim_{x \rightarrow 2} f(x) = \text{Does Not Exist}$ $\lim_{x \rightarrow 0} f(x) = -\frac{5}{2}$



2. $f(x) = \frac{2x^2 + 7}{x^2 + 5}$

Hole(s): None

x-int(s): None

y-int: $(0, \frac{7}{5})$

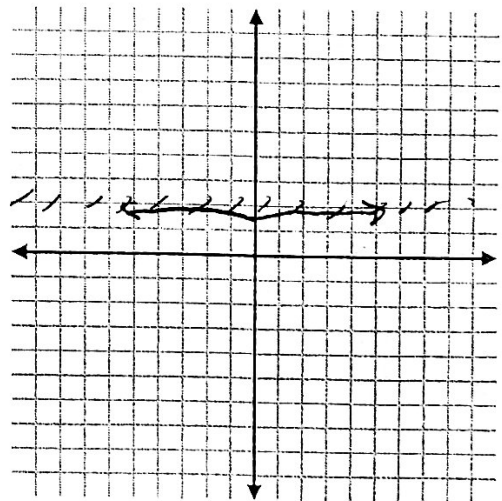
Equations of

ALL asymptotes: $y=2$

$\lim_{x \rightarrow -\infty} f(x) = 2$ $\lim_{x \rightarrow \infty} f(x) = 2$

$\lim_{x \rightarrow 0^-} f(x) = \frac{7}{5}$ $\lim_{x \rightarrow 0^+} f(x) = \frac{7}{5}$

$\lim_{x \rightarrow 0} f(x) = \frac{7}{5}$



3. $f(x) = \frac{x^3}{x^2 - 4}$

$$\begin{array}{r} x^2 + 0x - 4 \overline{) x^3 + 0x^2 + 0x + 0} \\ \underline{-(x^3 + 0x - 4x)} \\ 4x \end{array}$$

Hole(s): None

x-int(s): (0, 0)

y-int: (0, 0)

Equations of

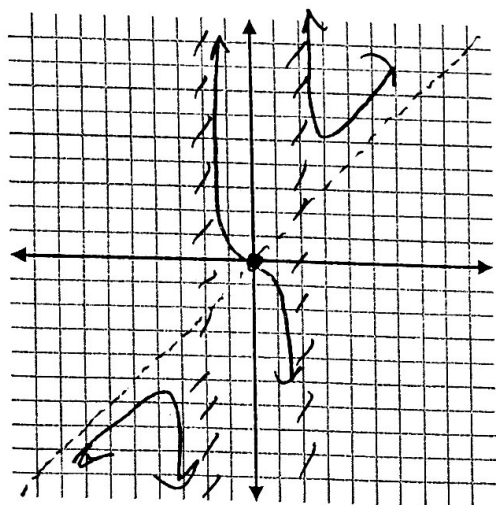
ALL asymptotes: $x=2$ $x=-2$ $y=x$

$\lim_{x \rightarrow -\infty} f(x) = -\infty$ $\lim_{x \rightarrow \infty} f(x) = \infty$

$\lim_{x \rightarrow -2^-} f(x) = -\infty$ $\lim_{x \rightarrow 2^-} f(x) = -\infty$

$\lim_{x \rightarrow -2^+} f(x) = \infty$ $\lim_{x \rightarrow 2^+} f(x) = \infty$

$\lim_{x \rightarrow -2} f(x) = \text{Does Not Exist}$ $\lim_{x \rightarrow 2} f(x) = \text{Does Not Exist}$



4. $f(x) = \frac{x-2}{x^2 - 2x - 8} = \frac{x-2}{(x-4)(x+2)}$

Hole(s): None

x-int(s): (2, 0)

y-int: (0, 1/4)

Equations of

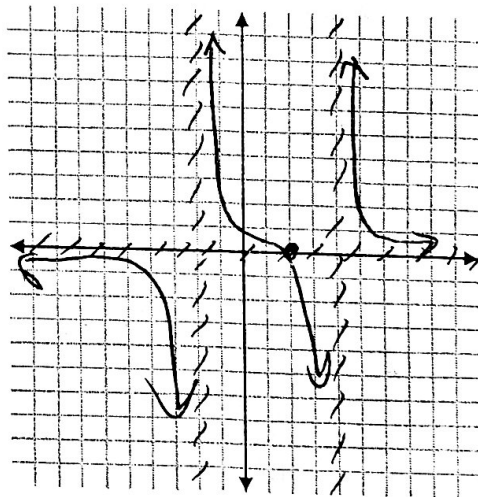
ALL asymptotes: $x=4$ $x=-2$ $y=0$

$\lim_{x \rightarrow -\infty} f(x) = 0$ $\lim_{x \rightarrow \infty} f(x) = 0$

$\lim_{x \rightarrow -2^-} f(x) = -\infty$ $\lim_{x \rightarrow 4^-} f(x) = -\infty$

$\lim_{x \rightarrow -2^+} f(x) = \infty$ $\lim_{x \rightarrow 4^+} f(x) = \infty$

$\lim_{x \rightarrow -2} f(x) = \text{Does Not Exist}$ $\lim_{x \rightarrow 4} f(x) = \text{Does Not Exist}$



5. $g(x) = \dots$

$$5. g(x) = \frac{x^2 - 5x - 6}{x^2 - x - 2} = \frac{(x-6)(x+1)}{(x-2)(x+1)}$$

Hole(s): $(-1, \frac{7}{3})$

x-int(s): $(6, 0)$

y-int: $(0, 3)$

Equations of

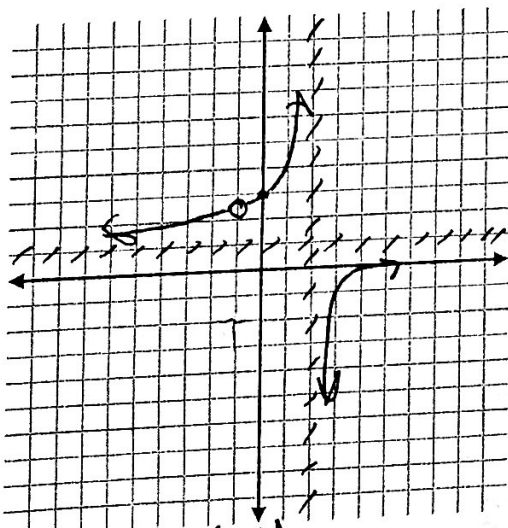
ALL asymptotes: $x=2$ $y=1$

$\lim_{x \rightarrow -\infty} f(x) = 1$ $\lim_{x \rightarrow \infty} f(x) = 1$

$\lim_{x \rightarrow -1^-} f(x) = \frac{7}{3}$ $\lim_{x \rightarrow 2^-} f(x) = \infty$

$\lim_{x \rightarrow -1^+} f(x) = \frac{7}{3}$ $\lim_{x \rightarrow 2^+} f(x) = -\infty$

$\lim_{x \rightarrow -1} f(x) = \frac{7}{3}$ $\lim_{x \rightarrow 2} f(x) = \text{Does Not Exist}$



Doesn't cross HA
 $1 = \frac{x-6}{x-2}$

$x-2 = x-6$
 $2 \neq -6$

$$6. f(x) = \frac{x+4}{-2x-6} = \frac{x+4}{-2(x+3)}$$

Hole(s): None

x-int(s): $(-4, 0)$

y-int: $(0, -2/3)$

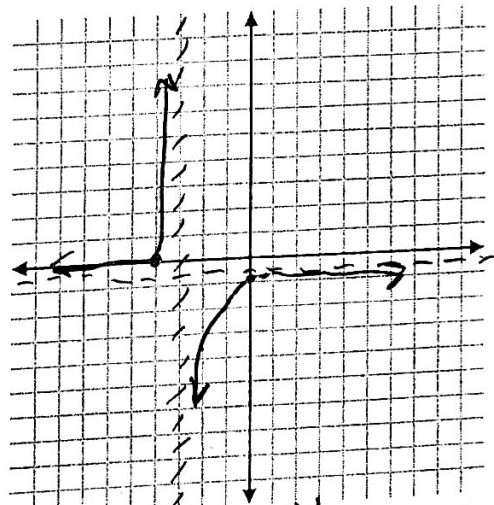
Equations of

ALL asymptotes: $x=-3$ $y=-\frac{1}{2}$

$\lim_{x \rightarrow -\infty} f(x) = -\frac{1}{2}$ $\lim_{x \rightarrow \infty} f(x) = \frac{1}{2}$

$\lim_{x \rightarrow -3^-} f(x) = \infty$ $\lim_{x \rightarrow -3^+} f(x) = -\infty$

$\lim_{x \rightarrow -3} f(x) = \text{Does Not Exist}$



Does Not Cross HA
 $(-2x-6)(-\frac{1}{2}) = \frac{x+4}{-2x-6}$

$x+3 = x+4$
 $3 \neq 4$